

Novel Symmetries in Christ-Lee Model

R. Kumar^{a,*} and A. Shukla^b

^a*S. N. Bose National Centre for Basic Sciences,
Salt Lake, Kolkata-700098, West Bengal, India*

^b*Indian Institute of Science Education and Research,
Mohanpur-741246, Kolkata, India*

E-mails: raviphynuc@gmail.com; as3756@iiserkol.ac.in

Abstract: We demonstrate that the gauge-fixed Lagrangian of the Christ-Lee model respects *four* fermionic symmetries, namely; (anti-)BRST symmetries, (anti-)co-BRST symmetries within the framework of BRST formalism. The appropriate anticommutators amongst the fermionic symmetries lead to a *unique* bosonic symmetry. It turns out that the algebra obeyed by the symmetry transformations (and their corresponding conserved charges) is reminiscent of the algebra satisfied by the de Rham cohomological operators of differential geometry. We also provide the physical realizations of the cohomological operators in terms of the symmetry properties. Thus, the present model provides a simple model for the Hodge theory.

PACS: 11.15.-q, 03.70.+k, 11.30.-j

Keywords: Christ-Lee model; (anti-)BRST symmetries; (anti-)co-BRST symmetries; de Rham cohomological operators; Hodge theory

*Present address: Department of Physics & Astrophysics, University of Delhi, Delhi-110007

1 Introduction

Symmetry principles play an important role to understand the laws of nature. The local gauge symmetry is one of the important symmetries in physics that relates the internal space with external spacetime. These symmetry transformations are generated due to the existence of the first-class constraints in a given theory [1, 2]. In order to quantize the system with first-class constraints, the Becchi-Rouet-Stora-Tyutin (BRST) formalism plays a decisive role [3–6]. In this formalism, for a local gauge symmetry associated with a given classical system, we have two linearly independent supersymmetric type BRST and anti-BRST symmetries at the quantum level. The geometrical interpretation and the origin of nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations have also been shown within the framework of superfield formalism [7–9].

The BRST formalism is one of the most institutive and elegant approaches to quantize the p -form ($p = 1, 2, 3, \dots$) gauge theories as well as it also encompasses many mathematical aspects in its fold. For instance, the mathematical aspects of BRST formalism have a deep connection with the abstract de Rham cohomological operators of differential geometry [10–12]. In our earlier work [13], we have established that any arbitrary Abelian p -form ($p = 1, 2, 3$) gauge theory in $D = 2p$ dimensions of spacetime, besides the usual (anti-)BRST symmetries, also endowed with the (anti-)co-BRST symmetries within the framework of BRST formalism. Thus, in $D = 2p$ dimensions of spacetime, we have been able to show that Abelian p -form theory provides a field-theoretic model for Hodge theory where the de Rham cohomological operators find their physical realizations in terms of the continuous symmetry transformations (and corresponding conserved charges). The discrete symmetry provides the analogue of the Hodge duality ($*$) operation. Recently, the rigid rotor as a toy model for the Hodge theory has also been illustrated in a great detail [14]. In a recent set of papers [15–17], it has been shown that the six continuous symmetries, present in a class of gauge field-theoretic models which happen to be models for the Hodge theory, lead to the same and unique canonical brackets amongst the creation and annihilation operators without taking any help of the canonical momenta. These canonical brackets play an important role in the quantization of the system.

It is worthwhile to mention that the BRST symmetries and related geometric approach have played a key role in our understanding of the quantization of superstring, supergravity, supersymmetric gauge theories and higher p -form ($p \geq 2$) gauge theories. Recently, the nilpotent BRST and anti-BRST symmetries for the perturbative quantum gravity have been established in (non-)linear gauges [18, 19]. Furthermore, the perturbative quantum gravity has also been studied in complex and noncommutative spacetime within the framework of BRST formalism [20–22]. The superspace formulation of higher derivative theories [23], Chern-Simons, Yang-Mills theories in the context of Batalin-Vilkovisky formalism [24, 25] and ABJM theory on deformed superspace [26, 27] have also been analyzed.

The Christ-Lee model is a classical constrained system which has been well-studied at the quantum level, too [28]. It is a gauge invariant model having two independent first-class constraints in the language of Dirac’s classification scheme of constraints. As far as the quantization of the Christ-Lee model is concerned, it has been studied in many different perspectives [29–32]. The consistent BRST quantization of this model is also carried out [33]. The purpose of our present investigation is two-fold. First, we explore the symmetries

associated with the Christ-Lee model. In the literature [33], only the off-shell nilpotent (anti-)BRST symmetries were known. In our present work, we show that, in addition to the usual (anti-)BRST symmetries, the nilpotent and absolutely anticommuting (anti-)co-BRST (also known as (anti-)dual-BRST) symmetries and a unique bosonic symmetry do exist for the present model. Second, the development of present model as a physical model for the Hodge theory is always interesting because of the fact that the abstract mathematical operators of differential geometry get their physical realizations in terms of the continuous symmetry transformations (and corresponding conserved charges). Thus, the Christ-Lee model provides a simple model for the Hodge theory within the framework of BRST formalism.

The contents of the present paper are as follows. In next section, we briefly discuss about the Christ-Lee model and associated local gauge symmetry. Third section is devoted for the discussion about the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations (and their corresponding charges). We show, in the fourth section, that the off-shell nilpotent and absolutely anticommuting (anti-)co-BRST symmetries also exist for the present model. In the fifth section, we discuss a unique bosonic symmetry and its corresponding generator. The bosonic symmetry emerges due to the existence of the non-vanishing anticommutators amongst the above fermionic symmetries. The sixth section is devoted for the discussion of the extended BRST algebra satisfied by the symmetry transformations (and corresponding conserved charges) and we also show that this algebra is reminiscent of the Hodge algebra obeyed by the de Rham cohomological operators of differential geometry. In this section, the physical realizations of the cohomological operators are also captured in terms of the symmetry properties. Finally, in the last section, we provide some concluding remarks and point out the future directions regarding our present investigation.

2 Christ-Lee model: local gauge symmetry

We begin with the Christ-Lee model in terms of the plane polar coordinates (r, θ) [28,30,33]. The first-order Lagrangian for the Christ-Lee model is given as

$$L_f = \dot{r} p_r + \dot{\theta} p_\theta - \frac{1}{2} p_r^2 - \frac{1}{2r^2} p_\theta^2 - z p_\theta - V(r), \quad (1)$$

where \dot{r} and $\dot{\theta}$ define the generalized velocities, z is another generalized coordinate, p_r and p_θ are the canonically conjugate momenta corresponding to the variables r and θ , respectively. $V(r)$ is the potential bounded from below and all the variables are function of time evolution parameter t . It is clear that the velocity \dot{z} is missing in the Lagrangian. As a consequence, the present system has a primary constraint:

$$\Omega_1 = \frac{\partial L_f}{\partial \dot{z}} = p_z \approx 0, \quad (2)$$

where p_z is the momentum corresponding to the auxiliary variable z . The symbol \approx defines weak equality in the sense of Dirac. The time evolution of the primary constraint Ω_1 leads

to the following secondary constraint:

$$\frac{d\Omega_1}{dt} = \frac{d}{dt} \left(\frac{\partial L_f}{\partial \dot{z}} \right) \approx 0 \Rightarrow \Omega_2 = p_\theta \approx 0. \quad (3)$$

It can be explicitly checked that the time evolution of Ω_2 does not yield any further constraint on the theory because Ω_2 commutes with the Hamiltonian of the present model. As a consequence, the Christ-Lee model endowed only with two first-class constraints in the Dirac's terminology [1, 2]. These first-class constraints are the generator of the gauge symmetry present in the system. The gauge symmetry generator can be written as [2, 34]

$$G = \dot{\chi} \Omega_1 + \chi \Omega_2, \quad (4)$$

where $\chi(t)$ is (time-dependent) local gauge parameter. Using the definition of a generator

$$\delta \xi(t) = -i [\xi(t), G], \quad (5)$$

where $\xi(t)$ denotes any generic variable present in the model, we obtain the following gauge transformations:

$$\delta z = \dot{\chi}(t), \quad \delta \theta = \chi(t), \quad \delta[r, p_r, p_\theta, V(r)]. \quad (6)$$

It is straightforward to check that under the above gauge transformations, the first-order Lagrangian (L_f) remains invariant (*i.e.*, $\delta L_f = 0$).

3 Off-shell nilpotent (anti-)BRST symmetry transformations and conserved charges

The gauge-fixed and (anti-)BRST invariant Lagrangian for the Christ-Lee model can be written as [33]

$$L = \dot{r} p_r + \dot{\theta} p_\theta - \frac{1}{2} p_r^2 - \frac{1}{2r^2} p_\theta^2 - z p_\theta - V(r) + \frac{1}{2} b^2 + b(\dot{z} + \theta) - i \dot{\bar{C}} \dot{C} + i \bar{C} C, \quad (7)$$

where the Nakanishi-Lautrup auxiliary variable b has been used to linearize the gauge-fixing term $-\frac{1}{2}(\dot{z} + \theta)^2$ as $\frac{b^2}{2} + b(\dot{z} + \theta)$. The anticommuting Faddeev-Popov ghost (C) and anti-ghost (\bar{C}) have ghost numbers $+1$ and -1 , respectively, whereas the remaining variables carry zero ghost number. The above Lagrangian respects the off-shell nilpotent ($s_{(a)b}^2 = 0$) and absolutely anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-)BRST transformations ($s_{(a)b}$). These transformations are

$$\begin{aligned} s_b z &= \dot{C}, & s_b \theta &= C, & s_b \bar{C} &= i b, & s_b[r, p_r, p_\theta, b, C] &= 0, \\ s_{ab} z &= \dot{\bar{C}}, & s_{ab} \theta &= \bar{C}, & s_{ab} C &= -i b, & s_{ab}[r, p_r, p_\theta, b, \bar{C}] &= 0. \end{aligned} \quad (8)$$

It can be explicitly checked that under the above off-shell nilpotent symmetry transformations, the Lagrangian L remains quasi-invariant. To be more precise, L transforms to a total time derivative under the (anti-)BRST transformations as follows:

$$s_b L = \frac{d}{dt} [b \dot{C}], \quad s_{ab} L = \frac{d}{dt} [b \dot{\bar{C}}]. \quad (9)$$

As a result, both BRST and anti-BRST transformations leave the action integral ($S = \int dt L$) invariant (*i.e.*, $s_{(a)b} S = 0$).

According to Noether's theorem, the invariance of the action under the (anti-)BRST transformations leads to the following conserved charges ($\dot{Q}_{(a)b} = 0$), namely

$$\begin{aligned} Q_b &= b \dot{C} + p_\theta C \equiv b \dot{C} - \dot{b} C, \\ Q_{ab} &= b \dot{\bar{C}} + p_\theta \bar{C} \equiv b \dot{\bar{C}} - \dot{b} \bar{C}. \end{aligned} \quad (10)$$

where on the r.h.s., we have used the equation of motion $p_\theta = -\dot{b}$. These charges are nilpotent of order two (*i.e.*, $Q_b^2 = 0$, $Q_{ab}^2 = 0$) and anticommuting (*i.e.*, $Q_b Q_{ab} + Q_{ab} Q_b = 0$) in nature. The conservation laws for the (anti-)BRST charges can be proven by using the following Euler-Lagrange equations of motion:

$$\begin{aligned} b &= -(\dot{z} + \theta), \quad \dot{b} + p_\theta = 0, \quad b = \dot{p}_\theta, \quad \dot{p}_r - \frac{p_\theta^2}{r^3} + V'(r) = 0, \\ \dot{r} &= p_r, \quad \dot{\theta} - z - \frac{p_\theta}{r^2} = 0, \quad \ddot{C} + C = 0, \quad \ddot{\bar{C}} + \bar{C} = 0. \end{aligned} \quad (11)$$

The above equations of motion have been derived from Lagrangian (7).

It turns out that the (anti-)BRST charges are the generators of the (anti-)BRST symmetry transformations, respectively. As one can readily check, the following relations are true, namely

$$s_b \Psi = -i[\Psi, Q_b]_\pm, \quad s_{ab} \Psi = -i[\Psi, Q_{ab}]_\pm, \quad (12)$$

where Ψ is any generic variable present in the Lagrangian (7). The (\pm) signs as the subscript on the square brackets correspond to (anti)commutator depending on the generic variable Ψ being (fermionic) bosonic in nature.

4 (Anti-)co-BRST Symmetries and their generators

We note that, in addition to the (anti-)BRST symmetries, the Lagrangian (7) also respects the following off-shell nilpotent (*i.e.*, $s_{(a)d}^2 = 0$) and absolutely anticommuting (*i.e.*, $s_d s_{ad} + s_d s_{ad} = 0$) (anti-)co-BRST (or (anti-)dual-BRST) symmetries ($s_{(a)d}$):

$$\begin{aligned} s_d z &= \bar{C}, & s_d \theta &= -\dot{\bar{C}}, & s_d C &= i p_\theta, & s_d [r, p_r, p_\theta, b, \bar{C}], \\ s_{ad} z &= C, & s_{ad} \theta &= -\dot{C}, & s_{ad} \bar{C} &= -i p_\theta, & s_{ad} [r, p_r, p_\theta, b, C]. \end{aligned} \quad (13)$$

Under the above nilpotent symmetry transformations, the Lagrangian L remains quasi-invariant, as one can check:

$$s_d L = -\frac{d}{dt}(p_\theta \dot{\bar{C}}), \quad s_{ad} L = -\frac{d}{dt}(p_\theta \dot{C}). \quad (14)$$

Thus, the action remains invariant (*i.e.*, $s_{(a)d} S = 0$) under the application of (anti-)co-BRST transformations. It is worthwhile to mention here that the total gauge fixing-term $\frac{b^2}{2} + b(\dot{z} + \theta)$ remains invariant under the off-shell nilpotent (anti-)co-BRST transformations.

The invariance of the action under the continuous (anti-)co-BRST transformations leads to the following conserved (anti-)co-BRST charges $Q_{(a)d}$:

$$\begin{aligned} Q_d &= b\bar{C} - p_\theta \dot{\bar{C}} \equiv b\bar{C} + \dot{b}\dot{\bar{C}}, \\ Q_{ad} &= bC - p_\theta \dot{C} \equiv bC + \dot{b}\dot{C}. \end{aligned} \quad (15)$$

The conservation laws ($\dot{Q}_{(a)d} = 0$) of the (anti-)co BRST charges can be proven by exploiting the Euler-Lagrange equations of motion (11). It turns out that these charges are the generator of the (anti-)dual-BRST symmetries as one can check that the following relations are true:

$$s_d \Psi = -i[\Phi, Q_d]_{\pm}, \quad s_{ad} \Phi = -i[\Psi, Q_{ad}]_{\pm}, \quad (16)$$

here Ψ represents the generic variable present in the model. The (\pm) signs as the subscript on the square brackets have same meaning as mentioned in our previous section. We note that the (anti-)co-BRST charges are also nilpotent ($Q_d^2 = Q_{ad}^2 = 0$) and anticommuting ($Q_d Q_{ad} + Q_{ad} Q_d = 0$) in nature.

5 Bosonic symmetry and conserved charge

Besides the above *four* fermionic symmetries as discussed in our earlier sections, we also have a *unique* bosonic symmetry present in the model. The bosonic symmetry (s_ω) is defined in terms of the fermionic symmetries as given below

$$s_\omega = \{s_b, s_d\} = -\{s_{ab}, s_{ad}\}. \quad (17)$$

It is worthwhile to point out that rest of the anticommutators amongst the fermionic symmetries are explicitly zero. For the sake of completeness, these anticommutators are as follows

$$\begin{aligned} \{s_b, s_{ab}\} &= 0, & \{s_b, s_{ad}\} &= 0, \\ \{s_d, s_{ab}\} &= 0, & \{s_d, s_{ad}\} &= 0. \end{aligned} \quad (18)$$

As a consequence, the above vanishing anticommutators do not define the symmetry. The bosonic symmetry transformations for all the variables are as follows:

$$s_\omega z = +i(b + \dot{p}_\theta), \quad s_\omega \theta = -i(\dot{b} - p_\theta), \quad s_\omega[r, p_r, p_\theta, b, C, \bar{C}] = 0. \quad (19)$$

Under the above bosonic symmetry, L transforms to a total time derivative as

$$s_\omega L = \frac{d}{dt} [i(b\dot{p}_\theta - \dot{b}p_\theta)]. \quad (20)$$

Thus, the action integral remains invariant. According to Noether's theorem, the above continuous bosonic symmetry leads to the derivation of the following conserved charge:

$$Q_\omega = i(b^2 + p_\theta^2) \equiv i(b\dot{p}_\theta - \dot{b}p_\theta). \quad (21)$$

The above charge is the generator of bosonic symmetry s_ω as one can check the following transformations $s_\omega \Psi = -i[\Psi, Q_\omega]$ is true for any generic variable Ψ .

6 Ghost scale and discrete symmetries

It is straightforward to check that the following ghost-scale symmetry transformations:

$$\begin{aligned} C &\rightarrow e^{+1\cdot\lambda} C, & \bar{C} &\rightarrow e^{-1\cdot\lambda} \bar{C}, \\ (r, \theta, z, p_r, p_\theta, b) &\rightarrow e^{0\cdot\lambda} (r, \theta, z, p_r, p_\theta, b), \end{aligned} \quad (22)$$

leave the Lagrangian (7) invariant. Here λ is a time independent (global) scale parameter. The numerals in the exponential represent the ghost number of the corresponding variables. In particular, the ghost numbers of $(C)\bar{C}$ are $(+1) - 1$, respectively whereas remaining variables have zero ghost number. Under the infinitesimal version of the above symmetry (with $\lambda = 1$)

$$s_g C = +C \quad s_g \bar{C} = -\bar{C}, \quad s_g [r, \theta, z, p_r, p_\theta, b] = 0, \quad (23)$$

the Lagrangian remains invariant (*i.e.*, $s_g L = 0$) and thus we obtain conserved ($\dot{Q}_g = 0$) ghost charge as follows:

$$Q_g = i(\bar{C} \dot{C} - \dot{\bar{C}} C). \quad (24)$$

It is straightforward to check that Q_g is the generator of the above infinitesimal ghost scale symmetry transformations. Besides the above continuous ghost scale symmetries, the ghost part of the Lagrangian (7) has the following discrete symmetries: $C \rightarrow \pm \bar{C}$ and $\bar{C} \rightarrow \mp C$. The latter symmetries play a decisive role in obtaining the anti-BRST and anti-co-BRST symmetries from the BRST and co-BRST symmetries, respectively. Further, we also note that under the discrete symmetry transformations, the ghost charge remains invariant.

7 Extended BRST algebra and cohomological aspects

Exploiting the operator form of the six independent continuous symmetries listed in equations (8), (13), (19) and (23), we obtain the following extended BRST algebra:

$$\begin{aligned} s_{(a)b}^2 &= 0, & s_{(a)d}^2 &= 0, & \{s_b, s_{ab}\} &= 0, \\ \{s_d, s_{ad}\} &= 0, & \{s_b, s_{ad}\} &= 0, & \{s_{ab}, s_d\} &= 0, \\ [s_g, s_b] &= s_b, & [s_g, s_{ab}] &= -s_{ab}, & [s_g, s_d] &= -s_d, \\ [s_g, s_{ad}] &= s_{ad}, & \{s_b, s_d\} &= -\{s_{ab}, s_{ad}\} = s_\omega, \\ [s_\omega, s_r] &= 0, & r &= b, ab, d, ad, g. \end{aligned} \quad (25)$$

It can also be checked that the above similar type of algebra is satisfied by the conserved charges as given below:

$$\begin{aligned} Q_{(a)b}^2 &= 0, & Q_{(a)d}^2 &= 0, & \{Q_b, Q_{ab}\} &= 0, \\ \{Q_d, Q_{ad}\} &= 0, & \{Q_b, Q_{ad}\} &= 0, & \{Q_d, Q_{ab}\} &= 0, \\ [Q_g, Q_b] &= -i Q_b, & [Q_g, Q_{ad}] &= -i Q_{ad}, \\ [Q_g, Q_{ab}] &= +i Q_{ab}, & [Q_g, Q_d] &= +i Q_d, \\ \{Q_b, Q_d\} &= -\{Q_{ab}, Q_{ad}\} = Q_\omega, \\ [Q_\omega, Q_r] &= 0, \end{aligned} \quad (26)$$

where, in proving the above algebra amongst the conserved charges, we have used the following basic (anti)commutators:

$$\begin{aligned} [r, p_r] &= i, & [\theta, p_\theta] &= i, & [z, b] &= i, \\ \{C, \dot{\bar{C}}\} &= 1, & \{\bar{C}, \dot{C}\} &= -1, \end{aligned} \quad (27)$$

and the remaining (anti)commutators are turn out to be zero. We point out that the algebra (26) can also be proven in a simple and straightforward manner by exploiting the definition of generator and the continuous symmetries (and corresponding charges). For instance, the nilpotency and anticommutativity properties of the fermionic charges can also be proved as follows:

$$\begin{aligned} s_b Q_b &= -i\{Q_b, Q_b\} = 0 \Rightarrow Q_b^2 = 0, \\ s_{ab} Q_{ab} &= -i\{Q_{ab}, Q_{ab}\} = 0 \Rightarrow Q_{ab}^2 = 0, \\ s_b Q_{ab} &= -i\{Q_{ab}, Q_b\} = 0 \Rightarrow Q_b Q_{ab} + Q_{ab} Q_b = 0, \\ s_{ab} Q_b &= -i\{Q_b, Q_{ab}\} = 0 \Rightarrow Q_b Q_{ab} + Q_{ab} Q_b = 0. \end{aligned} \quad (28)$$

Similarly, one can also compute other relations that appear in eq. (26).

It is to be noted that the algebra given in (25) and (26) are reminiscent of the Hodge algebra obeyed by de Rham cohomological operators (d, δ, Δ) of differential geometry. The later algebra is [35–39]

$$\begin{aligned} d^2 &= 0, & \delta^2 &= 0, & \{d, \delta\} &= \Delta = d\delta + \delta d, \\ [\Delta, d] &= 0, & [\Delta, \delta] &= 0. \end{aligned} \quad (29)$$

where $(\delta)d$ are the (co-)exterior derivatives and Δ is the Laplacian operator. A close look at the above algebra and the algebra obeyed by the continuous symmetry transformations (and corresponding charges) shows that there is a one-to-two mapping between the cohomological operators and the symmetry transformations (their corresponding conserved charges)

$$\begin{aligned} (s_b, s_{ad}) &\rightarrow d, & (s_{ab}, s_d) &\rightarrow \delta, \\ s_\omega &= \{s_b, s_d\} = -\{s_{ab}, s_{ad}\} \rightarrow \Delta, \\ (Q_b, Q_{ad}) &\rightarrow d, & (Q_{ab}, Q_d) &\rightarrow \delta, \\ Q_\omega &= \{Q_b, Q_d\} = -\{Q_{ab}, Q_{ad}\} \rightarrow \Delta. \end{aligned} \quad (30)$$

As a consequence, the continuous symmetry transformations (and their corresponding charges) provide the physical realizations of the abstract mathematical de Rham cohomological operators. Thus, the present Christ-Lee model provides a model for Hodge theory.

As far as the mathematical properties of the cohomological operators are concerned, we note that the exterior derivative d , when acts on any given n -form f_n of degree n , increases the degree of the form by one unit (*i.e.*, $df_n \sim f_{n+1}$) whereas the co-exterior derivative δ does its opposite when it operates on f_n (*i.e.*, $\delta f_n \sim f_{n-1}$). Further, the Laplacian operator Δ does not affect the degree of the form when it operates on a given form (*i.e.*, $\Delta f_n \sim f_n$).

These sacrosanct properties can be captured in terms of ghost number. If we look carefully on symmetry transformations (8), (13) and (17), we note that the sets (s_b, s_{ad}) and (s_{ab}, s_d) increase and decrease the ghost numbers by one, respectively when they operate on any generic variable while s_ω does not change the ghost number. We can also capture these properties in terms of the conserved charges. For this purpose, we define a state $|\psi\rangle_n$ (in the total quantum Hilbert space of states) as follows:

$$i Q_g |\psi\rangle_n = n |\psi\rangle_n, \quad (31)$$

where n defines the ghost number of the state $|\psi\rangle_n$ as the eigenvalue of the operator $i Q_g$. Using the algebra (26) and validity of the eq. (31), we yield the following relations:

$$\begin{aligned} i Q_g Q_b |\psi\rangle_n &= (n+1) Q_b |\psi\rangle_n, \\ i Q_g Q_{ad} |\psi\rangle_n &= (n+1) Q_{ad} |\psi\rangle_n, \\ i Q_g Q_d |\psi\rangle_n &= (n-1) Q_d |\psi\rangle_n, \\ i Q_g Q_{ab} |\psi\rangle_n &= (n-1) Q_{ab} |\psi\rangle_n, \\ i Q_g Q_\omega |\psi\rangle_n &= n Q_\omega |\psi\rangle_n. \end{aligned} \quad (32)$$

It evident that the states $Q_b |\psi\rangle_n$, $Q_d |\psi\rangle_n$ and $Q_\omega |\psi\rangle_n$ are endowed with the ghost numbers equal to $(n+1)$, $(n-1)$ and n , respectively. Similarly, the states $Q_{ad} |\psi\rangle_n$, $Q_{ab} |\psi\rangle_n$ and $Q_\omega |\psi\rangle_n$ have ghost numbers $(n+1)$, $(n-1)$ and n , respectively. Thus, the following sets (Q_b, Q_d, Q_ω) and $(Q_{ad}, Q_{ab}, -Q_\omega)$, as the conserved charges present in our model, provide the physical realizations of the de Rham cohomological operators (d, δ, Δ) of differential geometry.

We wrap up this section with the remarks that, in the realm of BRST quantization, the physicality criteria $Q_{(a)b} |phys\rangle = 0$ lead to the following requirements: $b |phys\rangle = 0$ and $\dot{b} |phys\rangle = 0$. Due to the validity of the Euler-Lagrange equations of motion (11), these requirements imply that the operator form of the first-class constraints $p_z \approx 0$ and $p_\theta \approx 0$ present in the original singular Lagrangian annihilate the physical state ($|phys\rangle$). In addition, it is interesting to note that the physicality conditions $Q_{(a)d} |phys\rangle = 0$ also produce the same requirements. Thus, the physicality criteria $Q_{(a)b} |phys\rangle = 0$ and $Q_{(a)d} |phys\rangle = 0$ are consistent with the Dirac's quantization of constrained system.

8 Conclusions

In our present investigation, we have discussed about the Christ-Lee model within the framework of BRST formalism. We have shown that, in addition to the nilpotent BRST and anti-BRST symmetries, there also exist *two* more nilpotent symmetries, namely; co-BRST and anti-co-BRST transformations (at quantum level). The non-vanishing anticommutators amongst the fermionic (anti-)BRST and (anti-)co-BRST symmetry transformations yield a unique bosonic symmetry. Furthermore, a continuous ghost-scale symmetry also exists for the present model. Thus, within the framework of BRST formalism, we have *six* continuous symmetry transformations. These continuous symmetries, according

to Noether's theorem, lead to the derivation of corresponding conserved charges. By exploiting the definition of a generator, we have also proven the nilpotency and anticommutativity properties of the fermionic charges in simpler way (cf. (28)).

The algebra satisfied by the operator form of the continuous symmetry transformations ($s_{(a)b}$, $s_{(a)d}$ and s_ω) is the replicate of the algebra obeyed by de Rham cohomological operators (*i.e.*, d , δ and Δ) of differential geometry (cf. (25) and (29)). The similar algebra is also satisfied by the conserved charges ($Q_{(a)b}$, $Q_{(a)d}$ and Q_ω) (cf. (26)). On the basis of the above, finally, we have shown that the de Rham cohomological operators find their physical meanings in terms of the continuous symmetry transformations (and corresponding conserved charges). In fact, we have two-to-one mapping between the symmetry transformations (and their corresponding generators) and the de Rham cohomological operators. Thus, the Christ-Lee model provides us a simple toy model for the Hodge theory within the framework of BRST formalism.

It would be a nice endeavor to derive the continuous transformations ($s_{(a)b}$, $s_{(a)d}$) by exploiting the celebrated supervariable approach where these transformations get their geometrical interpretation in terms of the Grassmannian derivatives. Furthermore, the invariance of Lagrangian and the nilpotency of charges can also be captured in terms of the supervariables and Grassmannian derivatives. It would also be interesting to derive the basic canonical brackets by exploiting the use of nilpotent charges within the framework of BRST formalism where one does not need any recourse of the canonical momenta. These are the some important issues for the present Christ-Lee model which we would like to address in our future publications elsewhere.

In our earlier works [11–17], we have shown that any arbitrary Abelian p -form ($p = 1, 2, 3$) gauge theory in $D = 2p$ dimensions of spacetime within the framework of BRST formalism and the $\mathcal{N} = 2$ SUSY quantum mechanical models turn out to be the tractable model for Hodge theory. It would be interesting to implement this idea in the case of a free particle system on a toric geometry [40], supersymmetric Yang-Mills [24, 41–43]. Furthermore, the derivation of proper (anti-)BRST symmetries with the help of superfield formalism would be a nice piece of work in the context of deformed super-Yang-Mills, supersymmetric Chern-Simons, ABJM and BLG theories [43–45].

Acknowledgments:

RK would like to thank UGC, Government of India, New Delhi, for financial support under the PDFSS scheme.

References

- [1] DIRAC P. A. M., *Lectures on Quantum Mechanics*, Belfer Graduate School of Science (Yeshiva University Press, New York, 1964).
- [2] SUNDERMEYER K., *Constrained Dynamics: Lecture Notes in Physics*, Vol. **169** (Springer, Berlin, 1982).

- [3] BECCHI C., ROUET A. and STORA R., *Phys. Lett. B*, **52**, (1974) 344.
- [4] BECCHI C., ROUET A. and STORA R., *Commun. Math. Phys.*, **42** (1975) 127.
- [5] BECCHI C., ROUET A. and STORA R., *Ann. Phys. (N.Y.)*, **98** (1976) 287.
- [6] TYUTIN I. V., Lebedev Institute Preprint, Report No: FIAN-39, 1975 (unpublished).
- [7] BONORA L. and TONIN M., *Phys. Lett. B*, **98** (1981) 48.
- [8] BONORA L., PASTI P. and TONIN M., *Nuovo Cimento A*, **63** (1981) 353.
- [9] DELBOURGO R. and JARVIS P. D., *J. Phys. A: Math. Gen.*, **15** (1981) 611.
- [10] MALIK R. P., *Int. J. Mod. Phys. A*, **22**, (2007) 3521.
- [11] GUPTA S. and MALIK R. P., *Eur. Phys. J. C*, **58** (2008) 517.
- [12] KUMAR R., KRISHNA S., SHUKLA A. and MALIK R. P., *Eur. Phys. J. C*, **72** (2012) 1980.
- [13] KUMAR R., KRISHNA S., SHUKLA A. and MALIK R. P., *Int. J. Mod. Phys. A*, **29** (2014) 1450135.
- [14] GUPTA S. and MALIK R. P., *Eur. Phys. J. C*, **68** (2010) 325.
- [15] KUMAR R., GUPTA S. and MALIK R. P., *Int. J. Theor. Phys.*, **55** (2016) 2857.
- [16] KUMAR R., GUPTA S. and MALIK R. P., *Commun. Theor. Phys.*, **61** (2014) 715.
- [17] SRINIVAS N., SHUKLA A. and MALIK R. P., *Int. J. Mod. Phys.*, **30** (2015) 1550166.
- [18] FAIZAL M., *Found. Phys.*, **41** (2011) 270.
- [19] FAIZAL M., *J. Phys. A*, **44** (2011) 402001.
- [20] FAIZAL M., *Phys. Lett. B*, **705** (2011) 120.
- [21] FAIZAL M., *Mod. Phys. Lett. A*, **27** (2012) 1250075.
- [22] FAIZAL M., *Mod. Phys. Lett. A*, **28** (2013) 1350034.
- [23] FAIZAL M. and KHAN M., *Eur. Phys. J. C*, **71** (2011) 1603.
- [24] FAIZAL M., *Commun. Theor. Phys.*, **58** (2012) 704.
- [25] FAIZAL M., *Int. J. Theor. Phys.*, **52** (2013) 392.
- [26] FAIZAL M., *Euro. Phys. Lett.*, **98** (2012) 31003.
- [27] FAIZAL M., *Phys. Rev. D*, **84** (2011) 105007.
- [28] CHRIST N. H. and LEE T. D., *Phys. Rev. D*, **49** (1980) 4.

- [29] COSTA M. E. V. and GIROTTI H. O., *Phys. Rev. D*, **24** (1981) 3323.
- [30] COSTA M. E. V., GIROTTI H. O. and SIMOES T. J. M.,
Phys. Rev. D, **32** (1985) 405.
- [31] BLAU S. K., *Ann. of Phys.*, **205** (1991) 392.
- [32] RABEI E. M., *Int. J. Theor. Phys.*, **42** (2003) 9.
- [33] KULSHRESHTHA U., KULSHRESHTHA D. S. and MULER-KIRSTEN H. J. W.,
Can. J. Phys., **73** (1995) 386.
- [34] ROTHE H. J. and ROTHE K. D., *Classical and Quantum Dynamics of Constrained Hamiltonian Systems* (World Scientific, Singapore, 2010).
- [35] NAKANISHI N. and OJIMA I., *Covariant Operator Formalism of Gauge Theory and Quantum Gravity* (World Scientific, Singapore, 1990).
- [36] EGUCHI T., GILKEY P. B. and HANSON A., *Phys. Rep.*, **66** (1980) 213.
- [37] MUKHI S. and MUKUNDA N., *Introduction to Topology, Differential Geometry and Group Theory for Physicists* (Wiley Eastern Private Limited, New Delhi, 1990).
- [38] NISHIJIMA K., *Prog. Theor. Phys.*, **80** (1988) 897.
- [39] DESER S., GOMBEROFF A., HENNEAUX M. and TEITELBOIM C.,
Phys. Lett. B, **400** (1997) 80.
- [40] KUMAR R., *Euro. Phys. Lett.*, **106** (2014) 51001.
- [41] WEINREB P. and FAIZAL M., *Phys. Lett. B*, **748** (2015) 102.
- [42] FAIZAL M., *Int. J. Geom. Meth. Mod. Phys.*, **12** (2015) 1550022.
- [43] FAIZAL M., *Int. J. Mod. Phys. A*, **28** (2013) 1350012.
- [44] FAIZAL M., *Mod. Phys. Lett. A*, **27** (2012) 1250147.
- [45] FAIZAL M., *JHEP*, **04** (2012) 017.